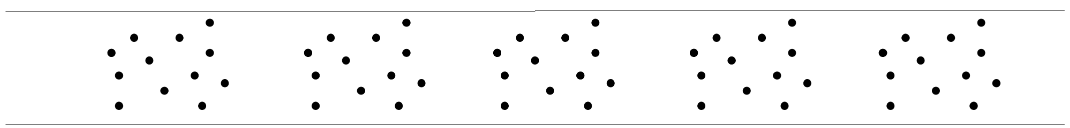
Tracking bunches of particles

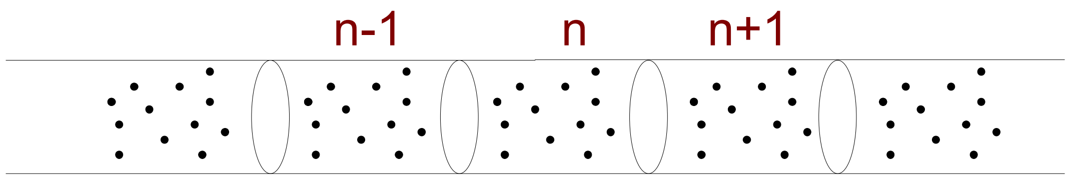
- Periodic longitudinal boundary conditions
- Characterizing distributions
  - moments
  - correlation coefficients
  - histograms
- Statistical definition of emittance
- Generating matched beams
  - including longitudinally uniform beams
- Compensating for statistical artifacts

Periodic longitudinal boundary conditions

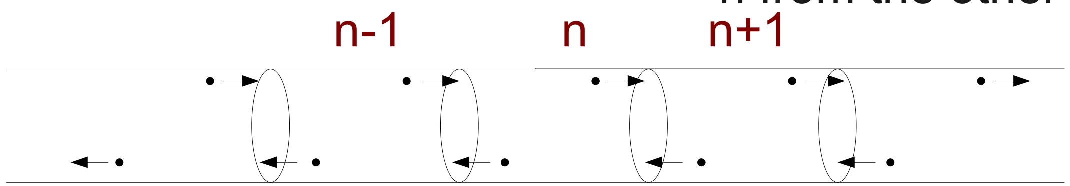
Take a long periodic (or uniform) beam



Look at period n



Each particle leaving n corresponds to another particle entering n from the other slide



Characterizing distributions

- If we are going to deal with large numbers of particles, we need to have a set of observables by which to characterize the distribution
- First moment (mean) $<x_i>$
- Second moment C:
- $$C_{ij} \equiv \langle (x_i - < x_i >)(x_j - < x_j >) \rangle$$
- Correlation coefficient R:

$$R_{ij} \equiv \frac{C_{ij}}{\sqrt{C_{ii} C_{jj}}}$$

characterizing, cont.

- These quantities are both useful in simulations and closely related to typical measuring devices
  - Beam position monitors report $\langle x_i \rangle$
  - Beam sizes are usually characterized by a width ~ sqrt(C_{ii})
- One-dimensional histograms are also useful and similar to the output of beam profile monitors.
- Of course, in the simulation we both have access to *and require* information not readily available in experiments.

Statistical definition of emittance

- $\epsilon = \sqrt{\det C_{ij}}$
  - note that this can be 2D (horizontal, vertical or longitudinal), 4D (transverse) or 6D (full phase space)
- Confusion danger: different conventions abound
  - This is “one sigma” ~ 36%
  - Unnormalized
- Numerical danger: taking determinants of even modestly-sized matrices is numerically tricky.

Generating matched beams

- Good news: real beams are usually well modeled by a Gaussian distribution in each coordinate
  - Gaussians are easy to generate. See, e.g., NR.
- Bad news: real beams contain correlations
  - Simplest case: 1D Courant-Snyder
  - $$R_{xx'} = \frac{\alpha}{\sqrt{1+\alpha^2}}$$
  - Beams with couplings between planes are much more complicated.

Aside: random number generators

- Computing (pseudo-) random numbers is difficult
- Canned routines can be bad
  - e.g., Intel Fortran ca 2002 (I haven't checked it lately)
- Ideally, you should always know which random algorithm you are using.
- You must always know how your random seed is being set.

Matching

- A matched beam has the same distribution after each period (turn, etc.)

$$x_f = M x_i$$

$$\sum x_f x_f^T = \sum M x_i x_i^T M^T$$

$$C \equiv \sum x_f x_f^T = \sum x_i x_i^T$$

$$C = M C M^T$$

$$M e = \lambda e \Rightarrow \{\lambda_i, e_i\}$$

$$E_i \equiv e_i e_i^\dagger$$

$$C = \sum_i a_i E_i$$

Given a set of constraints (widths, emittances, etc.), the problem is then to solve for the a_i

Subtleties

- Will the given procedure always work?
  - Only if the all degrees of freedom, including longitudinal, are "matchable"
    - i.e., stable RF
  - If the procedure fails, the components E_i will fail to span the space: $$det\left(\sum_i E_i\right) = 0$$
  -
- What if we have a uniform longitudinal beam or a bunching/debunching beam?
  - One possibility: only match transversely
    - Neglects to compensate for dispersion

Special case: uniform longitudinal beam

- If we are modeling a uniform longitudinal beam with periodic boundary conditions, the map does not tell the whole story.
  - And if we are not using periodic boundary conditions, the beam will not stay uniform!
  - If the longitudinal coordinate is uniform, symmetry prevents any couplings to other coordinates. All couplings to the spatial longitudinal coordinate (other than the self-coupling) should be set to zero in the map for the purposes of matching.

Fixing the distribution

- It is easy to generate a set of uncorrelated vectors {v} with Gaussian distribution with unit widths: $< v_i v_j >= \delta_{ij}$
- We want a set of vectors {r} with correlations given by the C from our procedure.
- Solution: $r_j = G_{jk} v_k$
- where $C = GG^T$
- GG^T is the Cholesky decomposition of C

Correcting for finite statistics

- In reality, our set of random vectors with have statistical fluctuations. These fluctuations are non-physical because we are typically using n macroparticles << N physical particles
- If we have $< v >= \bar{v}$
- and $< v_i v_j >= X_{ij}$
- then another Cholesky decomposition $X = HH^T$
- allows the solution $r_j = A_{jk}(v_k - \bar{v})$
- with $A = GH^{-1}$