Modern Computational Accelerator Physics

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Background for Accelerator Simulation Techniques

Lagrangian and Hamiltonian Dynamics

Disclaimer: the next few sections are mostly courtesy of Robert Ryne's lecture notes.

- Lorentz force equations. For a particle of charge q and mass m,

they are given in MKSA units by

$$\frac{d\vec{p}^{\mathrm{mech}}}{dt} = q\vec{E} + q\vec{v} \times \vec{B},$$

where $\vec{v}$ is the particle's velocity, and where $\vec{E}$ and $\vec{B}$ are the electric and magnetic field, respectively.

- The quantity $\vec{p}^{\mathrm{mech}}$ denotes the mechanical momentum,

which is given in Cartesian coordinates by

$$\vec{p}^{\mathrm{mech}} = \gamma m \vec{v},$$

Lagrangian and Hamiltonian Dynamics (2)

- For a charged particle in electromagnetic fields,

$$L = -mc^2\sqrt{1-\beta^2} - q\psi + q\vec{v}\cdot\vec{A},$$

where $\psi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ are the scalar and vector potentials, respectively.

- The canonical momentum,
p_i, is conjugate to the generalized coordinate, q_i,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}.$$

- The Hamiltonian $H(\vec{q}, \vec{p}, t)$ is related to the

Lagrangian, $L(\vec{q}, \dot{\vec{q}}, t)$ according to

$$H(\vec{q}, \vec{p}, t) = \sum_i p_i \dot{q}_i - L.$$

Lagrangian and Hamiltonian Dynamics (3)

- Hamilton's equations are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

t-Dependent vs. z-Dependent Dynamics

- Integrating by t through an accelerator turns out to be very inconvenient.
- Introduce

$$p_t = -H(\vec{q}, \vec{p}, t)$$

- Invert to obtain p_z as a function of (x, p_x, y, p_y, t, p_t) and z.
- Define

$$K(x, p_x, y, p_y, t, p_t; z) = -p_z.$$

t-Dependent vs. z-Dependent Dynamics (2)

- Then

$$x' = \frac{\partial K}{\partial p_x} \quad , \qquad p_x' = -\frac{\partial K}{\partial x}, \tag{1}$$

$$y' = \frac{\partial K}{\partial p_y} \quad , \qquad p_y' = -\frac{\partial K}{\partial y}, \tag{2}$$

$$t' = \frac{\partial K}{\partial p_t} \quad , \qquad p_t' = -\frac{\partial K}{\partial t}, \tag{3}$$

where a prime denotes d/dz.

- K is the new Hamiltonian with z as the independent variable.

$$\begin{aligned} K(x, p_x, y, p_y, t, p_t; z) \;&=\; -\left[(p_t + q\psi)^2/c^2 - m^2c^2 \right. \\ &\qquad \left. -(p_x - qA_x)^2 - (p_y - qA_y)^2\right]^{1/2} - qA_z. \end{aligned}$$