Modern Computational Accelerator Physics

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A symplectic integrator

Symplecticity

- Symplectic transformations are an important part of computation accelerator physics technology
  - We will define and discuss them later
- For now, we need to know three crucial facts
  - (1) Symplectic transformations preserve phase space
    - Allows long-term tracking
  - (2) The energy variation in a symplectic transformation is bounded
    - See previous
  - (3) In two dimensions, a transformation is symplectic iff

$$det J = 1.$$

In higher dimensions symplecticity implies $det J = 1$, but $det J = 1$ does *not* necessarily imply simplecticity.

A Symplectic Integrator

For the special case

$$\frac{d}{dt}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ f(y_0, t) \end{pmatrix},$$

We can write

$$y_0^{n+1} = y_0^n + h y_1^{n+1/2}$$
$$y_1^{n+1/2} = y_1^{n-1/2} + h f(y_0^n, t).$$

This is known as a second-order leapfrog method.

Symplecticity Check

Is the second-order leapfrog method symplectic?

$$J \equiv \left(\begin{array}{cc} \frac{\partial y_0^{n+1}}{\partial y_0^n} & \frac{\partial y_0^{n+1}}{\partial y_0^{n-1/2}} \\ \frac{\partial y_1^{n+1/2}}{\partial y_0^n} & \frac{\partial y_1^{n+1/2}}{\partial y_0^{n-1/2}} \end{array} \right)$$

$$J = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

So,

$$detJ = 1.$$

Yes, it is symplectic.

Assignment 2

- (1) Implement a second order leap frog integrator as a replacement for our GSL-based integrator. Use the approximation

$$y_1^n = \frac{1}{2}\left(y_1^{n-1/2} + y_1^{n+1/2}\right)$$

for the return value only.

- (2) Try a more interesting model: a pendulum. Create phase-space plots.
  - (a) Compare against GSL integrator.
    - rk2imp is useful in that it generates many steps.

In the pendulum case, there will be unbound orbits. I suggest using pyplot.axis([xmin, xmax, ymin, ymax])

Make sure you see unbound orbits.